

THEOREMS OF THE PROPOSITIONAL CALCULUS

EQUIVALENCE AND TRUE

- (3.1) **Axiom, Associativity of \equiv :** $((p \equiv q) \equiv r) \equiv (p \equiv (q \equiv r))$
- (3.2) **Axiom, Symmetry of \equiv :** $p \equiv q \equiv q \equiv p$
- (3.3) **Axiom, Identity of \equiv :** $true \equiv q \equiv q$
- (3.4) *true*
- (3.5) **Reflexivity of \equiv :** $p \equiv p$

NEGATION, INEQUIVALENCE, AND FALSE

- (3.8) **Axiom, Definition of *false*:** $false \equiv \neg true$
- (3.9) **Axiom, Distributivity of \neg over \equiv :** $\neg(p \equiv q) \equiv \neg p \equiv q$
- (3.10) **Axiom, Definition of \neq :** $(p \neq q) \equiv \neg(p \equiv q)$
- (3.11) $\neg p \equiv q \equiv p \equiv \neg q$
- (3.12) **Double negation:** $\neg\neg p \equiv p$
- (3.13) **Negation of *false*:** $\neg false \equiv true$
- (3.14) $(p \neq q) \equiv \neg p \equiv q$
- (3.15) $\neg p \equiv p \equiv false$
- (3.16) **Symmetry of \neq :** $(p \neq q) \equiv (q \neq p)$
- (3.17) **Associativity of \neq :** $((p \neq q) \neq r) \equiv (p \neq (q \neq r))$
- (3.18) **Mutual associativity:** $((p \neq q) \equiv r) \equiv (p \neq (q \equiv r))$
- (3.19) **Mutual interchangeability:** $p \neq q \equiv r \equiv p \equiv q \neq r$

DISJUNCTION

- (3.24) **Axiom, Symmetry of \vee :** $p \vee q \equiv q \vee p$
- (3.25) **Axiom, Associativity of \vee :** $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- (3.26) **Axiom, Idempotency of \vee :** $p \vee p \equiv p$
- (3.27) **Axiom, Distributivity of \vee over \equiv :** $p \vee (q \equiv r) \equiv p \vee q \equiv p \vee r$
- (3.28) **Axiom, Excluded Middle:** $p \vee \neg p$
- (3.29) **Zero of \vee :** $p \vee true \equiv true$
- (3.30) **Identity of \vee :** $p \vee false \equiv p$
- (3.31) **Distributivity of \vee over \vee :** $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$
- (3.32) $p \vee q \equiv p \vee \neg q \equiv p$

CONJUNCTION

- (3.35) **Axiom, Golden rule:** $p \wedge q \equiv p \equiv q \equiv p \vee q$
- (3.36) **Symmetry of \wedge :** $p \wedge q \equiv q \wedge p$

- (3.37) **Associativity of \wedge :** $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- (3.38) **Idempotency of \wedge :** $p \wedge p \equiv p$
- (3.39) **Identity of \wedge :** $p \wedge \text{true} \equiv p$
- (3.40) **Zero of \wedge :** $p \wedge \text{false} \equiv \text{false}$
- (3.41) **Distributivity of \wedge over \vee :** $p \wedge (q \vee r) \equiv (p \wedge q) \wedge (p \wedge r)$
- (3.42) **Contradiction:** $p \wedge \neg p \equiv \text{false}$
- (3.43) **Absorption:** (a) $p \wedge (p \vee q) \equiv p$
 (b) $p \vee (p \wedge q) \equiv p$
- (3.44) **Absorption:** (a) $p \wedge (\neg p \vee q) \equiv p \wedge q$
 (b) $p \vee (\neg p \wedge q) \equiv p \vee q$
- (3.45) **Distributivity of \vee over \wedge :** $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- (3.46) **Distributivity of \wedge over \vee :** $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- (3.47) **De Morgan:** (a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$
 (b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- (3.48) $p \wedge q \equiv p \wedge \neg q \equiv \neg p$
- (3.49) $p \wedge (q \equiv r) \equiv p \wedge q \equiv p \wedge r \equiv p$
- (3.50) $p \wedge (q \equiv p) \equiv p \wedge q$
- (3.51) **Replacement:** $(p \equiv q) \wedge (r \equiv p) \equiv (p \equiv q) \wedge (r \equiv q)$
- (3.52) **Definition of \equiv :** $p \equiv q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
- (3.53) **Exclusive or:** $p \neq q \equiv (\neg p \wedge q) \vee (p \wedge \neg q)$
- (3.55) $(p \wedge q) \wedge r \equiv p \equiv q \equiv r \equiv p \vee q \equiv q \vee r \equiv r \vee p \equiv p \vee q \vee r$

IMPLICATION

- (3.57) **Axiom, Definition of Implication:** $p \Rightarrow q \equiv p \vee q \equiv q$
- (3.58) **Axiom, Consequence:** $p \Leftarrow q \equiv q \Rightarrow p$
- (3.59) **Definition of implication:** $p \Rightarrow q \equiv \neg p \vee q$
- (3.60) **Definition of implication:** $p \Rightarrow q \equiv p \wedge q \equiv p$
- (3.61) **Contrapositive:** $p \Rightarrow q \equiv \neg q \Rightarrow \neg p$
- (3.62) $p \Rightarrow (q \equiv r) \equiv p \wedge q \equiv p \wedge r$
- (3.63) **Distributivity of \Rightarrow over \equiv :** $p \Rightarrow (q \equiv r) \equiv p \Rightarrow q \equiv p \Rightarrow r$
- (3.64) $p \Rightarrow (q \Rightarrow r) \equiv (p \Rightarrow q) \Rightarrow (p \Rightarrow r)$
- (3.65) **Shunting:** $p \wedge q \Rightarrow r \equiv p \Rightarrow (q \Rightarrow r)$
- (3.66) $p \wedge (p \Rightarrow q) \equiv p \wedge q$
- (3.67) $p \wedge (q \Rightarrow p) \equiv p$
- (3.68) $p \vee (p \Rightarrow q) \equiv \text{true}$
- (3.69) $p \vee (q \Rightarrow p) \equiv q \Rightarrow p$

- (3.70) $p \vee q \Rightarrow p \wedge q \equiv p \equiv q$
- (3.71) **Reflexivity of \Rightarrow :** $p \Rightarrow p \equiv true$
- (3.72) **Right zero of \Rightarrow :** $p \Rightarrow true \equiv true$
- (3.73) **Left identity of \Rightarrow :** $true \Rightarrow p \equiv p$
- (3.74) $p \Rightarrow false \equiv \neg p$
- (3.75) $false \Rightarrow p \equiv true$
- (3.76) **Weakening/strengthening:** (a) $p \Rightarrow p \vee q$
 (b) $p \wedge q \Rightarrow p$
 (c) $p \wedge q \Rightarrow p \vee q$
 (d) $p \vee (q \wedge r) \Rightarrow p \vee q$
 (e) $p \wedge q \Rightarrow p \wedge (q \vee r)$
- (3.77) **Modus ponens:** $p \wedge (p \Rightarrow q) \Rightarrow q$
- (3.78) $(p \Rightarrow r) \wedge (q \Rightarrow r) \equiv (p \vee q \Rightarrow r)$
- (3.79) $(p \Rightarrow r) \wedge (\neg p \Rightarrow r) \equiv r$
- (3.80) **Mutual implication:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \equiv (p \equiv q)$
- (3.81) **Antisymmetry:** $(p \Rightarrow q) \wedge (q \Rightarrow p) \Rightarrow (p \equiv q)$
- (3.82) **Transitivity:** (a) $(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
 (b) $(p \equiv q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
 (c) $(p \Rightarrow q) \wedge (q \equiv r) \Rightarrow (p \Rightarrow r)$

LEIBNIZ AS AN AXIOM

- (3.83) **Axiom, Leibniz:** $e = f \Rightarrow E_e^z = E_f^z$
- (3.84) **Substitution:** (a) $(e = f) \wedge E_e^z \equiv (e = f) \wedge E_f^z$
 (b) $(e = f) \Rightarrow E_e^z \equiv (e = f) \Rightarrow E_f^z$
 (c) $q \wedge (e = f) \Rightarrow E_e^z \equiv q \wedge (e = f) \Rightarrow E_f^z$
- (3.85) **Replace by true:** (a) $p \Rightarrow E_p^z \equiv p \Rightarrow E_{true}^z$
 (b) $q \wedge p \Rightarrow E_p^z \equiv q \wedge p \Rightarrow E_{true}^z$
- (3.86) **Replace by false:** (a) $E_p^z \Rightarrow p \equiv E_{false}^z \Rightarrow p$
 (b) $E_p^z \Rightarrow p \vee q \equiv E_{false}^z \Rightarrow p \vee q$
- (3.87) **Replace by true:** $p \wedge E_p^z \equiv p \wedge E_{true}^z$
- (3.88) **Replace by false:** $p \vee E_p^z \equiv p \vee E_{false}^z$
- (3.89) **Shannon:** $E_p^z \equiv (p \wedge E_{true}^z) \vee (\neg p \wedge E_{false}^z)$
- (4.1) $p \Rightarrow (q \Rightarrow p)$
- (4.2) **Monotonicity of \vee :** $(p \Rightarrow q) \Rightarrow (p \vee r \Rightarrow q \vee r)$
- (4.3) **Monotonicity of \wedge :** $(p \Rightarrow q) \Rightarrow (p \wedge r \Rightarrow q \wedge r)$

PROOF TECHNIQUES

- (4.4) **Deduction:** To prove $P \Rightarrow Q$, assume P and prove Q .
- (4.5) **Case analysis:** If E_{true}^z , E_{false}^z are theorems, then so is E_P^z .
- (4.6) **Case analysis:** $(p \vee q \vee r) \wedge (p \Rightarrow s) \wedge (q \Rightarrow s) \wedge (r \Rightarrow s) \Rightarrow s$
- (4.7) **Mutual implication:** To prove $P \equiv Q$, prove $P \Rightarrow Q$ and $Q \Rightarrow P$.
- (4.9) **Proof by contradiction:** To prove P , prove $\neg P \Rightarrow false$.
- (4.12) **Proof by contrapositive:** To prove $P \Rightarrow Q$, prove $\neg Q \Rightarrow \neg P$

GENERAL LAWS OF QUANTIFICATION

For symmetric and associative binary operator \star with identity u .

- (8.13) **Axiom, Empty range:** $(\star x \mid false : P) = u$
- (8.14) **Axiom, One-point rule:** Provided $\neg occurs('x', 'E')$,
 $(\star x \mid x = E : P) = P[x := E]$
- (8.15) **Axiom, Distributivity:** Provided $P, Q: \mathbb{B}$ or R is finite,
 $(\star x \mid R : P) \star (\star x \mid R : Q) = (\star x \mid R : P \star Q)$
- (8.16) **Axiom, Range split:** Provided $R \wedge S \equiv false$ and
 $P: \mathbb{B}$ or R and S are finite,
 $(\star x \mid R \vee S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$
- (8.17) **Axiom, Range split:** Provided $P: \mathbb{B}$ or R and S finite,
 $(\star x \mid R \vee S : P) \star (\star x \mid R \wedge S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$
- (8.18) **Axiom, Range split for idempotent \star :**
 $(\star x \mid R \vee S : P) = (\star x \mid R : P) \star (\star x \mid S : P)$
- (8.19) **Axiom, Interchange of dummies:** Provided \star is idempotent or
 R, Q are finite, $\neg occurs('y', 'R')$, and $\neg occurs('x', 'Q')$,
 $(\star x \mid R : (\star y \mid Q : P)) = (\star y \mid Q : (\star x \mid R : P))$
- (8.20) **Axiom, Nesting:** Provided $\neg occurs('y', 'R')$,
 $(\star x, y \mid R \wedge Q : P) = (\star x \mid R : (\star y \mid Q : P))$
- (8.21) **Axiom, Dummy renaming:** Provided $\neg occurs('y', 'R, P')$,
 $(\star x \mid R : P) = (\star y \mid R[x := y] : P[x := y])$
- (8.22) **Change of dummy:** Provided $\neg occurs('y', 'R, P')$, and f
has an inverse, $(\star x \mid R : P) = (\star y \mid R[x := f.y] : P[x := f.y])$
- (8.23) **Split off term:** $(\star i \mid 0 \leq i < n + 1 : P) = (\star i \mid 0 \leq i < n : P) \star P_n^i$

THEOREMS OF THE PREDICATE CALCULUS

UNIVERSAL QUANTIFICATION

- (9.2) **Axiom, Trading:** $(\forall x \mid R : P) \equiv (\forall x \mid R \Rightarrow P)$
- (9.3) **Trading:** (a) $(\forall x \mid R : P) \equiv (\forall x \mid \neg R \vee P)$
(b) $(\forall x \mid R : P) \equiv (\forall x \mid R \wedge P \equiv R)$
(c) $(\forall x \mid R : P) \equiv (\forall x \mid R \vee P \equiv P)$

- (9.4) **Trading:** (a) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \Rightarrow P)$
 (b) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : \neg R \vee P)$
 (c) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \wedge P \equiv R)$
 (d) $(\forall x \mid Q \wedge R : P) \equiv (\forall x \mid Q : R \vee P \equiv P)$
- (9.5) **Axiom, Distributivity of \vee over \forall :** Prov. $\neg\text{occurs}(x', 'P')$,
 $P \vee (\forall x \mid R : Q) \equiv (\forall x \mid R : P \vee Q)$
- (9.6) Provided $\neg\text{occurs}(x', 'P')$, $(\forall x \mid R : P) \equiv P \vee (\forall x \mid R : \neg R)$
- (9.7) **Distributivity of \wedge over \forall :** Provided $\neg\text{occurs}(x', 'P')$,
 $\neg(\forall x \mid R : \neg R) \Rightarrow ((\forall x \mid R : P \wedge Q) \equiv P \wedge (\forall x \mid R : Q))$
- (9.8) $(\forall x \mid R : \text{true}) \equiv \text{true}$
- (9.9) $(\forall x \mid R : P \equiv Q) \Rightarrow ((\forall x \mid R : P) \equiv (\forall x \mid R : Q))$
- (9.10) **Range weakening/strengthening:** $(\forall x \mid Q \vee R : P) \Rightarrow (\forall x \mid Q : P)$
- (9.11) **Body weakening/strengthening:** $(\forall x \mid R : P \wedge Q) \Rightarrow (\forall x \mid R : P)$
- (9.12) **Monotonicity of \forall :**
 $(\forall x \mid R : Q \Rightarrow P) \Rightarrow ((\forall x \mid R : Q) \Rightarrow (\forall x \mid R : P))$
- (9.13) **Instantiation:** $(\forall x \mid R : P) \Rightarrow P[x := e]$
- (9.16) P is a theorem iff $(\forall x \mid R : P)$ is a theorem.

EXISTENTIAL QUANTIFICATION

- (9.17) **Axiom, Generalized De Morgan:**
 $(\exists x \mid R : P) \equiv \neg(\forall x \mid R : \neg P)$
- (9.18) **Generalized De Morgan:** (a) $\neg(\exists x \mid R : \neg P) \equiv (\forall x \mid R : P)$
 (b) $\neg(\exists x \mid R : P) \equiv (\forall x \mid R : \neg P)$
 (c) $(\exists x \mid R : \neg P) \equiv \neg(\forall x \mid R : P)$
- (9.19) **Trading:** $(\exists x \mid R : P) \equiv (\exists x \mid R : P)$
- (9.20) **Trading:** $(\exists x \mid Q \wedge R : P) \equiv (\exists x \mid Q : R \wedge P)$
- (9.21) **Distributivity of \wedge over \exists :** Provided $\neg\text{occurs}(x', 'P')$,
 $P \wedge (\exists x \mid R : Q) \equiv (\exists x \mid R : P \wedge Q)$
- (9.22) Provided $\neg\text{occurs}(x', 'P')$, $(\exists x \mid R : P) \equiv P \wedge (\exists x \mid R : R)$
- (9.23) **Distributivity of \vee over \exists :** Provided $\neg\text{occurs}(x', 'P')$,
 $(\exists x \mid R : P) \Rightarrow ((\exists x \mid R : P \vee Q) \equiv P \vee (\exists x \mid R : Q))$
- (9.24) $(\exists x \mid R : \text{false}) \equiv \text{false}$
- (9.25) **Range weakening/strengthening:** $(\exists x \mid R : P) \Rightarrow (\exists x \mid Q \vee R : P)$
- (9.26) **Body weakening/strengthening:** $(\exists x \mid R : P) \Rightarrow (\exists x \mid R : P \vee Q)$
- (9.27) **Monotonicity of \exists :**
 $(\forall x \mid R : Q \Rightarrow P) \Rightarrow ((\exists x \mid R : Q) \Rightarrow (\exists x \mid R : P))$
- (9.28) **\exists -Introduction:** $P[x := E] \Rightarrow (\exists x \mid R : P)$
- (9.29) **Interchange of quantifications:**
 Provided $\neg\text{occurs}(y', 'R')$ and $\neg\text{occurs}(x', 'Q')$,
 $(\exists x \mid R : (\forall y \mid Q : P)) \Rightarrow (\forall y \mid Q : (\exists x \mid R : P))$
- (9.30) Provided $\neg\text{occurs}(x', 'Q')$,
 $(\exists x \mid R : P) \Rightarrow Q$ is a theorem iff $(R \wedge P)[x := \hat{x}] \Rightarrow Q$ is a theorem